

SCIENCE AND HYPOTHESIS



PART I.

NUMBER AND MAGNITUDE.

CHAPTER I.

ON THE NATURE OF MATHEMATICAL REASONING.

I.

THE very possibility of mathematical science seems an insoluble contradiction. If this science is only deductive in appearance, from whence is derived that perfect rigour which is challenged by none? If, on the contrary, all the propositions which it enunciates may be derived in order by the rules of formal logic, how is it that mathematics is not reduced to a gigantic tautology? The syllogism can teach us nothing essentially new, and if everything must spring from the principle of identity, then everything should be capable of being reduced to that principle. Are we then to admit that the enunciations of all

the theorems with which so many volumes are filled, are only indirect ways of saying that A is A?

No doubt we may refer back to axioms which are at the source of all these reasonings. If it is felt that they cannot be reduced to the principle of contradiction, if we decline to see in them any more than experimental facts which have no part or lot in mathematical necessity, there is still one resource left to us: we may class them among *à priori* synthetic views. But this is no solution of the difficulty—it is merely giving it a name; and even if the nature of the synthetic views had no longer for us any mystery, the contradiction would not have disappeared; it would have only been shirked. Syllogistic reasoning remains incapable of adding anything to the data that are given it; the data are reduced to axioms, and that is all we should find in the conclusions.

No theorem can be new unless a new axiom intervenes in its demonstration; reasoning can only give us immediately evident truths borrowed from direct intuition; it would only be an intermediary parasite. Should we not therefore have reason for asking if the syllogistic apparatus serves only to disguise what we have borrowed?

The contradiction will strike us the more if we open any book on mathematics; on every page the author an-

nounces his intention of generalising some proposition already known. Does the mathematical method proceed from the particular to the general, and, if so, how can it be called deductive?

Finally, if the science of number were merely analytical, or could be analytically derived from a few synthetic intuitions, it seems that a sufficiently powerful mind could with a single glance perceive all its truths; nay, one might even hope that some day a language would be invented simple enough for these truths to be made evident to any person of ordinary intelligence.

Even if these consequences are challenged, it must be granted that mathematical reasoning has of itself a kind of creative virtue, and is therefore to be distinguished from the syllogism. The difference must be profound. We shall not, for instance, find the key to the mystery in the frequent use of the rule by which the same uniform operation applied to two equal numbers will give identical results. All these modes of reasoning, whether or not reducible to the syllogism, properly so called, retain the analytical character, and *ipso facto*, lose their power.

II.

The argument is an old one. Let us see how Leibnitz tried to show that two and two make four. I assume the number one to be defined, and also the operation $x + 1$ —*i.e.*, the adding of unity to a given number x . These definitions, whatever they may be, do not enter into the subsequent reasoning. I next define the numbers 2, 3, 4 by the equalities

$$(1) \quad 1 + 1 = 2; \quad (2) \quad 2 + 1 = 3; \quad (3) \quad 3 + 1 = 4;$$

and in the same way I define the operation $x + 2$ by the relation

$$(4) \quad x + 2 = (x + 1) + 1.$$

Given this, we have

$$\begin{aligned} 2 + 2 &= (2 + 1) + 1, && \text{(def. 4);} \\ (2 + 1) + 1 &= 3 + 1, && \text{(def. 2);} \\ 3 + 1 &= 4, && \text{(def. 3);} \\ \text{whence } 2 + 2 &= 4, && \text{Q.E.D.} \end{aligned}$$

It cannot be denied that this reasoning is purely analytical. But if we ask a mathematician, he will reply: "This is not a demonstration properly so called; it is a

verification.” We have confined ourselves to bringing together one or other of two purely conventional definitions, and we have verified their identity; nothing new has been learned. *Verification* differs from proof precisely because it is analytical, and because it leads to nothing. It leads to nothing because the conclusion is nothing but the premisses translated into another language. A real proof, on the other hand, is fruitful, because the conclusion is in a sense more general than the premisses. The equality $2 + 2 = 4$ can be verified because it is particular. Each individual enunciation in mathematics may be always verified in the same way. But if mathematics could be reduced to a series of such verifications it would not be a science. A chess-player, for instance, does not create a science by winning a piece. There is no science but the science of the general. It may even be said that the object of the exact sciences is to dispense with these direct verifications.

III.

Let us now see the geometer at work, and try to surprise some of his methods. The task is not without difficulty; it is not enough to open a book at random and to anal-

use any proof we may come across. First of all, geometry must be excluded, or the question becomes complicated by difficult problems relating to the rôle of the postulates, the nature and the origin of the idea of space. For analogous reasons we cannot avail ourselves of the infinitesimal calculus. We must seek mathematical thought where it has remained pure—*i. e.*, in Arithmetic. But we still have to choose; in the higher parts of the theory of numbers the primitive mathematical ideas have already undergone so profound an elaboration that it becomes difficult to analyse them.

It is therefore at the beginning of Arithmetic that we must expect to find the explanation we seek; but it happens that it is precisely in the proofs of the most elementary theorems that the authors of classic treatises have displayed the least precision and rigour. We may not impute this to them as a crime; they have obeyed a necessity. Beginners are not prepared for real mathematical rigour; they would see in it nothing but empty, tedious subtleties. It would be waste of time to try to make them more exacting; they have to pass rapidly and without stopping over the road which was trodden slowly by the founders of the science.

Why is so long a preparation necessary to habituate

oneself to this perfect rigour, which it would seem should naturally be imposed on all minds? This is a logical and psychological problem which is well worthy of study. But we shall not dwell on it; it is foreign to our subject. All I wish to insist on is, that we shall fail in our purpose unless we reconstruct the proofs of the elementary theorems, and give them, not the rough form in which they are left so as not to weary the beginner, but the form which will satisfy the skilled geometer.

DEFINITION OF ADDITION.

I assume that the operation $x + 1$ has been defined; it consists in adding the number 1 to a given number x . Whatever may be said of this definition, it does not enter into the subsequent reasoning.

We now have to define the operation $x + a$, which consists in adding the number a to any given number x . Suppose that we have defined the operation

$$x + (a - 1);$$

the operation $x + a$ will be defined by the equality

$$(1) \quad x + a = [x + (a - 1)] + 1.$$

We shall know what $x+a$ is when we know what $x+(a-1)$ is, and as I have assumed that to start with we know what $x+1$ is, we can define successively and “by recurrence” the operations $x+2$, $x+3$, etc. This definition deserves a moment’s attention; it is of a particular nature which distinguishes it even at this stage from the purely logical definition; the equality (1), in fact, contains an infinite number of distinct definitions, each having only one meaning when we know the meaning of its predecessor.

PROPERTIES OF ADDITION.

Associative.—I say that

$$a + (b + c) = (a + b) + c;$$

in fact, the theorem is true for $c = 1$. It may then be written

$$a + (b + 1) = (a + b) + 1;$$

which, remembering the difference of notation, is nothing but the equality (1) by which I have just defined addition. Assume the theorem true for $c = n$, I say that it will be true for $c = n + 1$. Let

$$(a + b) + n = a + (b + n);$$

it follows that

$$[(a + b) + c] + 1 = [a + (b + c)] + 1;$$

or by def. (1),

$$(a + b) + (c + 1) = a + (b + c + 1) = a + [b + (c + 1)];$$

which shows by a series of purely analytical deductions that the theorem is true for $c + 1$. Being true for $c = 1$, we see that it is successively true for $c = 2$, $c = 3$, etc.

Commutative.—(1) I say that

$$a + 1 = 1 + a.$$

The theorem is evidently true for $a = 1$; we can *verify* by purely analytical reasoning that if it is true for $a = c$ it will be true for $a = c + 1$.¹ Now, it is true for $a = 1$, and therefore is true for $a = 2$, $a = 3$, and so on. This is what is meant by saying that the proof is demonstrated “by recurrence.”

(2) I say that

$$a + b = b + a.$$

¹For $(\gamma + 1) + 1 = (1 + \gamma) + 1 = 1 + (\gamma + 1)$.—[TR.]

The theorem has just been shown to hold good for $b = 1$, and it may be verified analytically that if it is true for $b = n$, it will be true for $b = n + 1$. The proposition is thus established by recurrence.

DEFINITION OF MULTIPLICATION.

We shall define multiplication by the equalities

$$(1) \quad a \times 1 = a;$$

$$(2) \quad a \times b = [a \times (b - 1)] + a.$$

Both of these include an infinite number of definitions; having defined $a \times 1$, it enables us to define in succession $a \times 2$, $a \times 3$, and so on.

PROPERTIES OF MULTIPLICATION.

Distributive.—I say that

$$(a + b) \times c = (a \times c) + (b \times c).$$

We can verify analytically that the theorem is true for $c = 1$; then if it is true for $c = n$, it will be true for $c = n + 1$. The proposition is then proved by recurrence.

Commutative.—(1) I say that

$$a \times 1 = 1 \times a.$$

The theorem is obvious for $a = 1$. We can verify analytically that if it is true for $a = \alpha$, it will be true for $a = \alpha + 1$.

(2) I say that

$$a \times b = b \times a.$$

The theorem has just been proved for $b = 1$. We can verify analytically that if it be true for $b = \alpha$ it will be true for $b = \alpha + 1$.

IV.

This monotonous series of reasonings may now be laid aside; but their very monotony brings vividly to light the process, which is uniform, and is met again at every step. The process is proof by recurrence. We first show that a theorem is true for $n = 1$; we then show that if it is true for $n - 1$ it is true for n , and we conclude that it is true for all integers. We have now seen how it may be used for the proof of the rules of addition and multiplication—that is to say, for the rules of the algebraical calculus.

This calculus is an instrument of transformation which lends itself to many more different combinations than the simple syllogism; but it is still a purely analytical instrument, and is incapable of teaching us anything new. If mathematics had no other instrument, it would immediately be arrested in its development; but it has recourse anew to the same process—*i.e.*, to reasoning by recurrence, and it can continue its forward march. Then if we look carefully, we find this mode of reasoning at every step, either under the simple form which we have just given to it, or under a more or less modified form. It is therefore mathematical reasoning *par excellence*, and we must examine it closer.

V.

The essential characteristic of reasoning by recurrence is that it contains, condensed, so to speak, in a single formula, an infinite number of syllogisms. We shall see this more clearly if we enunciate the syllogisms one after another. They follow one another, if one may use the expression, in a cascade. The following are the hypothetical syllogisms:—The theorem is true of the number 1. Now, if it is true of 1, it is true of 2; therefore it is true of 2.

Now, if it is true of 2, it is true of 3; hence it is true of 3, and so on. We see that the conclusion of each syllogism serves as the minor of its successor. Further, the majors of all our syllogisms may be reduced to a single form. If the theorem is true of $n - 1$, it is true of n .

We see, then, that in reasoning by recurrence we confine ourselves to the enunciation of the minor of the first syllogism, and the general formula which contains as particular cases all the majors. This unending series of syllogisms is thus reduced to a phrase of a few lines.

It is now easy to understand why every particular consequence of a theorem may, as I have above explained, be verified by purely analytical processes. If, instead of proving that our theorem is true for all numbers, we only wish to show that it is true for the number 6 for instance, it will be enough to establish the first five syllogisms in our cascade. We shall require 9 if we wish to prove it for the number 10; for a greater number we shall require more still; but however great the number may be we shall always reach it, and the analytical verification will always be possible. But however far we went we should never reach the general theorem applicable to all numbers, which alone is the object of science. To reach it we should require an infinite number of syllogisms, and we

should have to cross an abyss which the patience of the analyst, restricted to the resources of formal logic, will never succeed in crossing.

I asked at the outset why we cannot conceive of a mind powerful enough to see at a glance the whole body of mathematical truth. The answer is now easy. A chess-player can combine for four or five moves ahead; but, however extraordinary a player he may be, he cannot prepare for more than a finite number of moves. If he applies his faculties to Arithmetic, he cannot conceive its general truths by direct intuition alone; to prove even the smallest theorem he must use reasoning by recurrence, for that is the only instrument which enables us to pass from the finite to the infinite. This instrument is always useful, for it enables us to leap over as many stages as we wish; it frees us from the necessity of long, tedious, and monotonous verifications which would rapidly become impracticable. Then when we take in hand the general theorem it becomes indispensable, for otherwise we should ever be approaching the analytical verification without ever actually reaching it. In this domain of Arithmetic we may think ourselves very far from the infinitesimal analysis, but the idea of mathematical infinity is already playing a preponderating part, and without it

there would be no science at all, because there would be nothing general.

VI.

The views upon which reasoning by recurrence is based may be exhibited in other forms; we may say, for instance, that in any finite collection of different integers there is always one which is smaller than any other. We may readily pass from one enunciation to another, and thus give ourselves the illusion of having proved that reasoning by recurrence is legitimate. But we shall always be brought to a full stop—we shall always come to an indemonstrable axiom, which will at bottom be but the proposition we had to prove translated into another language. We cannot therefore escape the conclusion that the rule of reasoning by recurrence is irreducible to the principle of contradiction. Nor can the rule come to us from experiment. Experiment may teach us that the rule is true for the first ten or the first hundred numbers, for instance; it will not bring us to the indefinite series of numbers, but only to a more or less long, but always limited, portion of the series.

Now, if that were all that is in question, the principle

of contradiction would be sufficient, it would always enable us to develop as many syllogisms as we wished. It is only when it is a question of a single formula to embrace an infinite number of syllogisms that this principle breaks down, and there, too, experiment is powerless to aid. This rule, inaccessible to analytical proof and to experiment, is the exact type of the *à priori* synthetic intuition. On the other hand, we cannot see in it a convention as in the case of the postulates of geometry.

Why then is this view imposed upon us with such an irresistible weight of evidence? It is because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it, and thereby of becoming conscious of it.

But it will be said, if the legitimacy of reasoning by recurrence cannot be established by experiment alone, is it so with experiment aided by induction? We see successively that a theorem is true of the number 1, of the number 2, of the number 3, and so on—the law is manifest, we say, and it is so on the same ground that every physical law is true which is based on a very large

but limited number of observations.

It cannot escape our notice that here is a striking analogy with the usual processes of induction. But an essential difference exists. Induction applied to the physical sciences is always uncertain, because it is based on the belief in a general order of the universe, an order which is external to us. Mathematical induction—*i.e.*, proof by recurrence—is, on the contrary, necessarily imposed on us, because it is only the affirmation of a property of the mind itself.

VII.

Mathematicians, as I have said before, always endeavour to generalise the propositions they have obtained. To seek no further example, we have just shown the equality

$$a + 1 = 1 + a,$$

and we then used it to establish the equality

$$a + b = b + a,$$

which is obviously more general. Mathematics may, therefore, like the other sciences, proceed from the particular to the general. This is a fact which might

otherwise have appeared incomprehensible to us at the beginning of this study, but which has no longer anything mysterious about it, since we have ascertained the analogies between proof by recurrence and ordinary induction.

No doubt mathematical recurrent reasoning and physical inductive reasoning are based on different foundations, but they move in parallel lines and in the same direction—namely, from the particular to the general.

Let us examine the case a little more closely. To prove the equality

$$(1) \quad a + 2 = 2 + a,$$

we need only apply the rule

$$a + 1 = 1 + a$$

twice, and write

$$(2) \quad a + 2 = a + 1 + 1 = 1 + a + 1 = 1 + 1 + a = 2 + a.$$

The equality thus deduced by purely analytical means is not, however, a simple particular case. It is something quite different. We may not therefore even say in the really analytical and deductive part of mathematical reasoning that we proceed from the general to the particular

in the ordinary sense of the words. The two sides of the equality (2) are merely more complicated combinations than the two sides of the equality (1), and analysis only serves to separate the elements which enter into these combinations and to study their relations.

Mathematicians therefore proceed “by construction,” they “construct” more complicated combinations. When they analyse these combinations, these aggregates, so to speak, into their primitive elements, they see the relations of the elements and deduce the relations of the aggregates themselves. The process is purely analytical, but it is not a passing from the general to the particular, for the aggregates obviously cannot be regarded as more particular than their elements.

Great importance has been rightly attached to this process of “construction,” and some claim to see in it the necessary and sufficient condition of the progress of the exact sciences. Necessary, no doubt, but not sufficient! For a construction to be useful and not mere waste of mental effort, for it to serve as a stepping-stone to higher things, it must first of all possess a kind of unity enabling us to see something more than the juxtaposition of its elements. Or more accurately, there must be some advantage in considering the construction rather than the

elements themselves. What can this advantage be? Why reason on a polygon, for instance, which is always decomposable into triangles, and not on elementary triangles? It is because there are properties of polygons of any number of sides, and they can be immediately applied to any particular kind of polygon. In most cases it is only after long efforts that those properties can be discovered, by directly studying the relations of elementary triangles. If the quadrilateral is anything more than the juxtaposition of two triangles, it is because it is of the polygon type.

A construction only becomes interesting when it can be placed side by side with other analogous constructions for forming species of the same genus. To do this we must necessarily go back from the particular to the general, ascending one or more steps. The analytical process "by construction" does not compel us to descend, but it leaves us at the same level. We can only ascend by mathematical induction, for from it alone can we learn something new. Without the aid of this induction, which in certain respects differs from, but is as fruitful as, physical induction, construction would be powerless to create science.

Let me observe, in conclusion, that this induction is only possible if the same operation can be repeated indefinitely. That is why the theory of chess can never become

a science, for the different moves of the same piece are limited and do not resemble each other.